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The C^α regularity of a class of non-homogeneous ultraparabolic equations

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Abstract

We obtain the C^α regularity for weak solutions of a class of non-homogeneous ultraparabolic equation, with measurable coefficients. The result generalizes our recent C^α regularity results of homogeneous ultraparabolic equations.

keywords: Non-homogeneous, ultraparabolic equations, C^α regularity

1 Introduction

The regularity of ultraparabolic equation becomes important since it has many applications. From mathematical points of view, it has some special algebraic structures and is degenerated. Though there are more and more studies on this problem in recent years, it is still unclear in general, whether the interior C^α regularity results hold for weak solutions of the ultraparabolic equations with bounded measurable coefficients like the parabolic cases.

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One of the typical example of the ultraparabolic equation is the following equation

$$\frac{\partial u}{\partial t} + y\frac{\partial u}{\partial x} - u^2\frac{\partial^2 u}{\partial y^2} = 0. \tag{1.1}$$

This is of strong degenerated parabolic type equations, more precisely, an ultraparabolic type equation. However, if the coefficient is smooth it satisfies the well known Hörmander's hypoellipticity conditions, which sheds lights on the smoothness of weak solutions. It is interesting if the weak solution of equation (1.1) is still smooth when the coefficient is only measurable functions.

On the other hand, the equation (1.1), if consider it as an equation of $\frac{1}{u}$, has the divergent form. A recent paper by Pascucci and Polidoro [12], Cinti, Pascucci and Polidoro [2] proved that the Moser iterative method still works for a class of ultraparabolic equations with measurable coefficients. Their results show that for a non-negative sub-solution u of (1.1), the L^∞ norm of u is bounded by the L^p norm ($p \geq 1$). This is a very important step to the final regularity of solutions of the ultraparabolic equations.

We seems to have proved in [15], [17] that the weak solution obtained in [14] of (1.1) is of C^α class, then u is smooth. In this paper, we are concerned with the C^α regularity of solutions of more general ultraparabolic equations.

We consider a class of non-homogeneous Kolmogorov-Fokker-Planck type operator on R^{N+1}:

$$Lu \equiv \sum_{i,j=1}^{m_0} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j} u) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} u - \partial_t u = 0, \tag{1.2}$$

where $(x,t) \in R^{N+1}$, $1 \leq m_0 \leq N$, and b_{ij} is constant for every $i,j = 1,\cdots,N$. Let $A = (a_{ij})_{N\times N}$, where $a_{ij} = 0$, if $i > m_0$ or $j > m_0$. We make the following assumptions on the coefficients of L:

(H_1) $a_{ij} = a_{ji} \in L^\infty(R^{N+1})$ and there exists a $\lambda > 0$ such that

$$\frac{1}{\lambda}\sum_{i=1}^{m_0}\xi_i^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \le \lambda\sum_{i=1}^{m_0}\xi_i^2$$

for every $(x,t) \in R^{N+1}$, and $\xi \in R^{m_0}$.

(H_2) The matrix $B = (b_{ij})_{N\times N}$ has the form

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_d \\ * & * & * & \cdots & * \end{pmatrix}$$

where B_k is a matrix $m_{k-1} \times m_k$ with rank m_k and $m_0 \ge m_1 \ge \cdots \ge m_d$, $m_0 + m_1 + \cdots + m_d = N$.

The requirements of matrix B in (H_2) ensure that the operator L with the constant a_{ij} satisfies the well-known Hörmander's hypoellipticity condition. We let λ satisfies $||B|| \le \lambda$ where the norm $||\cdot||$ is in the sense of matrix norm. We refer [2] for more details on non-homogeneous Kolmogorov-Fokker-Planck type operator on R^{N+1}.

The Schauder type estimate of (1.2) has been obtained for example, in [18], [19] and [16]. Besides, the regularity of weak solutions have been studied by Bramanti, Cerutti and Manfredini [1], Polidoro and Ragusa [13] assuming a weak continuity on the coefficient a_{ij}. It is quite interesting whether the weak solution has Hölder regularity under the assumption (H_1) on a_{ij}. One of the approach to the Hölder estimates is to obtain the Harnack type inequality. In the case of elliptic equations with measurable coefficients, the Harnack inequality is obtained by J. Moser [9] via an estimate of BMO functions due to F. John and L. Nirenberg together with the Moser iteration method. J. Moser [10] also obtained the Harnack inequality for parabolic equations

with measurable coefficients by generalizing the John-Nirenberg estimates to the parabolic case. Another approach to the Hölder estimates is given by S. N. Kruzhkov [6], [8] based on the Moser iteration to obtain a local priori estimates, which provides a short proof for the parabolic equations. Nash [11] introduced another technique relying on the Poincaré inequality and obtained the Hölder regularity. Also De Giorgi developed an approach to obtain the Hölder regularity for elliptic equations.

We prove a Poincaré type inequality for non-negative weak sub-solutions of (1.2). Then we apply it to obtain a local priori estimates which implies the Hölder estimates for ultraparabolic equation (1.2).

Let D_{m_0} be the gradient with respect to the variables $x_1, x_2, \cdots, x_{m_0}$. And

$$Y = \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t.$$

We say that u is a *weak solution* if it satisfies (1.2) in the distribution sense, that is for any $\phi \in C_0^\infty(\Omega)$, where Ω is a open subset of R^{N+1}, then

$$\int_\Omega \phi Y u - (Du)^T A D\phi = 0, \tag{1.3}$$

and u, $D_{m_0}u$, $Yu \in L^2_{\text{loc}}(\Omega)$.

Our main result is the following theorem.

Theorem 1.1 *Under the assumptions* (H_1) *and* (H_2)*, the weak solution of (1.2) is Hölder continuous.*

2 Some Preliminary Results

One of the important feature of equation (1.2) is that the fundamental solution can be written explicitly if the coefficients a_{ij} is constant (cf. [4], [7]).

Besides, there are some geometric and algebraic structures in the space R^{N+1} induced by the constant matrix B (see for instance, [7]).

We follow the earlier notations and give some basic properties used for example, by [2] and [7], and more details see [2] and [7].

Let $E(\tau) = \exp(-\tau \mathrm{B}^{\mathrm{T}})$. For $(x,t), (\xi,\tau) \in R^{N+1}$, set

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau),$$

then $(R^{N+1}, \circ)$ is a Lie group with identity element $(0,0)$, and the inverse of an element is $(x,t)^{-1} = (-E(-t)x, -t)$. The left translation by (ξ,τ) given by

$$(x,t) \mapsto (\xi,\tau) \circ (x,t),$$

is a invariant translation to operator L when coefficient a_{ij} is constant. The associated dilation to operator L with constant coefficient a_{ij} is given by

$$\delta_t = diag(tI_{m_0}, t^3 I_{m_1}, \cdots, t^{2d+1} I_{m_d}, t^2),$$

where I_{m_k} denotes the $m_k \times m_k$ identity matrix, t is a positive parameter, also we assume

$$D_t = diag(tI_{m_0}, t^3 I_{m_1}, \cdots, t^{2d+1} I_{m_d}),$$

and denote

$$Q = m_0 + 3m_1 + \cdots + (2d+1)m_d,$$

then the number $Q+2$ is usually called the homogeneous dimension of $(R^{N+1}, \circ)$ with respect to the dilation δ_t.

The norm in R^{N+1}, related to the group of translations and dilation to the equation is defined by

$$||(x,t)|| = r,$$

if r is the unique positive solution to the equation

$$\frac{x_1^2}{r^{2\alpha_1}} + \frac{x_2^2}{r^{2\alpha_2}} + \cdots + \frac{x_N^2}{r^{2\alpha_N}} + \frac{t^2}{r^4} = 1,$$

where $(x,t) \in R^{N+1} \setminus \{0\}$ and

$$\alpha_1 = \cdots = \alpha_{m_0} = 1, \quad \alpha_{m_0+1} = \cdots = \alpha_{m_0+m_1} = 3, \cdots,$$

$$\alpha_{m_0+\cdots+m_{d-1}+1} = \cdots = \alpha_N = 2d+1.$$

And $||(0,0)|| = 0$. The balls at a point (x_0, t_0) is defined by

$$\mathcal{B}_r(x_0,t_0) = \{(x,t) | \quad ||(x_0,t_0)^{-1} \circ (x,t)|| \le r\},$$

and

$$\mathcal{B}_r^-(x_0,t_0) = \mathcal{B}_r(x_0,t_0) \cap \{t < t_0\}.$$

For convenience, we sometimes use the cube replace the balls. The cube at point $(0,0)$ is given by

$$\mathcal{C}_r(0,0) = \{(x,t) | \quad |t| \le r^2, \quad |x_1| \le r^{\alpha_1}, \cdots, |x_N| \le r^{\alpha_N}\}.$$

It is easy to see that there exists a constant Λ such that

$$\mathcal{C}_{\frac{r}{\Lambda}}(0,0) \subset \mathcal{B}_r(0,0) \subset \mathcal{C}_{\Lambda r}(0,0),$$

where Λ only depends on B and N.

When the matrix $(a_{ij})_{N\times N}$ is of constant matrix, we denoted it by A_0, and A_0 has the form

$$A_0 = \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix}$$

then let

$$\mathcal{C}(t) \equiv \int_0^t E(s) A_0 E^T(s) ds,$$

which is positive when $t > 0$, and the operator L_1 takes the form

$$L_1 = div(A_0 D) + Y,$$

whose fundamental solution $\Gamma_1(\cdot, \zeta)$ with pole in $\zeta \in R^{N+1}$ has been constructed as follows:

$$\Gamma_1(z,\zeta) = \Gamma_1(\zeta^{-1} \circ z, 0), \qquad z, \zeta \in R^{N+1}, \quad z \ne \zeta,$$

where $z=(x,t)$. And $\Gamma_1(z,0)$ can be written down explicitly

$$\Gamma_1(z,0)=\begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det \mathcal{C}(t)}}\exp(-\frac{1}{4}\langle \mathcal{C}^{-1}(t)x,x\rangle - t\,tr(B)) & \text{if} \quad t>0, \\ 0 & \text{if} \quad t\le 0. \end{cases} \tag{2.1}$$

There are some basic estimates for Γ_1 (see [2])

$$\Gamma_1(z,\zeta)\le C||\zeta^{-1}\circ z||^{-Q}, \tag{2.2}$$

$$|\partial_{\xi_i}\Gamma_1(z,\zeta)|\le C||\zeta^{-1}\circ z||^{-Q-1}, \tag{2.3}$$

where $i=1,\cdots,m_0$, for all $z,\zeta\in R^N\times(0,T]$.

A weak sub-solution of (1.2) in a domain Ω is a function u such that u, $D_{m_0}u$, $Yu\in L^2_{loc}(\Omega)$ and for any $\phi\in C_0^\infty(\Omega)$, $\phi\ge 0$,

$$\int_\Omega \phi Yu-(Du)^TAD\phi\ge 0. \tag{2.4}$$

Similarly, let $Y_0=<x,\,B_0D>-\partial_t$, where B_0 has the form

$$\begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_d \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

We denote $L_0=div(A_0D)+Y_0$, and can define in the same way $E_0(t)$, $\mathcal{C}_0(t)$, and $\Gamma_0(z,\zeta)$ with respect to B_0. We recall that $\mathcal{C}_0(t)(t>0)$ (see[7]) satisfies

$$\mathcal{C}_0(t)=D_{t^{\frac{1}{2}}}\mathcal{C}_0(1)D_{t^{\frac{1}{2}}}. \tag{2.5}$$

The following lemma is obtained by Lanconelli and Polidoro (see [7]), which is need in our proof.

Lemma 2.1 *In addition to the above assumptions, for every given $T > 0$, there exist positive constants C_T and C'_T such that*

$$\langle \mathcal{C}_0(t)x, x\rangle(1 - C_T\, t) \le \langle \mathcal{C}(t)x, x\rangle \le \langle \mathcal{C}_0(t)x, x\rangle(1 + C_T\, t), \tag{2.6}$$

$$\langle \mathcal{C}_0^{-1}(t)x, x\rangle(1 - C_T\, t) \le \langle \mathcal{C}^{-1}(t)x, x\rangle \le \langle \mathcal{C}_0^{-1}(t)x, x\rangle(1 + C_T\, t), \tag{2.7}$$

$$C_T'^{-1} t^Q(1 - C_T\, t) \le \det\mathcal{C}(t) \le C'_T\, t^Q(1 + C_T\, t), \tag{2.8}$$

for every $(x, t) \in R^N \times (0, T]$ and $t < \frac{1}{C_T}$.

A result of Cinti, Pascucci and Polidoro obtained by using the Moser's iterative method (see [2]) states as follows.

Lemma 2.2 *Let u be a non-negative weak sub-solution of (1.2) in Ω. Let $(x_0, t_0) \in \Omega$ and $\overline{\mathcal{B}_r^-(x_0, t_0)} \subset \Omega$ and $p \ge 1$. Then there exists a positive constant C which depends only on λ and Q such that, for $0 < r \le 1$*

$$\sup_{\mathcal{B}^-_{\frac{r}{2}}(x_0,t_0)} u^p \le \frac{C}{r^{Q+2}} \int_{\mathcal{B}^-_r(x_0,t_0)} u^p, \tag{2.9}$$

provided that the last integral converges.

We copy a classical potential estimates (cf. (1.11) in [3]) here to prove the Poincaré type inequality.

Lemma 2.3 *Let $(R^{N+1}, \circ)$ is a homogeneous Lie group of homogeneous dimension $Q + 2$, $\alpha \in (0, Q + 2)$ and $G \in C(R^{N+1} \setminus \{0\})$ be a δ_λ-homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(R^{N+1})$ for some $p \in (1, \infty)$, then*

$$G_f(z) \equiv \int_{R^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta,$$

is defined almost everywhere and there exists a constant $C = C(Q,p)$ such that

$$||G_f||_{L^q(R^{N+1})} \leq C \max_{||z||=1} |G(z)| \quad ||f||_{L^p(R^{N+1})}, \tag{2.10}$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2}.$$

Corollary 2.1 *Let $f \in L^2(R^{N+1})$, recall the definitions in [2]*

$$\Gamma_1(f)(z) = \int_{R^{N+1}} \Gamma_1(z,\zeta) f(\zeta) d\zeta, \qquad \forall z \in R^{N+1},$$

and

$$\Gamma_1(D_{m_0} f)(z) = -\int_{R^{N+1}} D_{m_0}^{(\zeta)} \Gamma_1(z,\zeta) f(\zeta) d\zeta, \qquad \forall z \in R^{N+1},$$

then exists a positive constant $C = C(Q,T,B)$ such that

$$\|\Gamma_1(f)\|_{L^{2\tilde{k}}(S_T)} \leq C \|f\|_{L^2(S_T)}, \tag{2.11}$$

and

$$\|\Gamma_1(D_{m_0} f)\|_{L^{2k}(S_T)} \leq C \|f\|_{L^2(S_T)}, \tag{2.12}$$

where $\tilde{k} = 1 + \frac{4}{Q-2}$, $k = 1 + \frac{2}{Q}$ and $S_T = R^N \times]0,T]$.

3 Proof of Main Theorem

To obtain a local estimates of solutions of the equation (1.2), for instance, at point (x_0, t_0), we may consider the estimates at a ball centered at $(0,0)$, since the equation (1.2) is invariant under the left group translation when a_{ij} is constant. By introducing a Poincaré type inequality, we prove the following Lemma 3.5 which is essential in the oscillation estimates in Kruzhkov's

approaches in parabolic case. Then the C^α regularity result follows easily by the standard arguments.

For convenience, in the following discussion, we let $x' = (x_1, \cdots, x_{m_0})$ and $x = (x', \overline{x})$. We consider the estimates in the following cube, instead of $\mathcal{B}_r^-$,

$$\mathcal{C}_r^- = \{(x,t) | \quad -r^2 \le t \le 0, |x'| \le r, |x_{m_0+1}| \le (\lambda N^2 r)^3, \cdots, |x_N| \le (\lambda N^2 r)^{2d+1}\}.$$

Let

$$K_r = \{x' | \quad |x'| \le r\},$$

$$S_r = \{\overline{x} \quad | \quad |x_{m_0+1}| \le (\lambda N^2 r)^3, \cdots, |x_N| \le (\lambda N^2 r)^{2d+1}\}.$$

Let $0 < \alpha, \beta < 1$ be constants, for fixed t and h, let

$$\mathcal{N}_{t,h} = \{(x', \overline{x}) \in K_{\beta r} \times S_{\beta r}, \quad u(\cdot, t) \ge h\}.$$

In the following discussions, we sometimes abuse the notations of $\mathcal{B}_r^-$ and $\mathcal{C}_r^-$, since there are equivalent, and we always assume $r \ll 1$ and $\lambda > 8$ in the following arguments, since λ can choose a large constant. Moreover, all constants depend on m_0, d or Q will be denoted by dependence on B.

Lemma 3.1 *Suppose that $u(x,t) \ge 0$ be a solution of equation (1.2) in $\mathcal{B}_r^-$ centered at $(0,0)$ and*

$$mes\{(x,t) \in \mathcal{B}_r^-, \quad u \ge 1\} \ge \frac{1}{2} mes(\mathcal{B}_r^-).$$

Then there exist constants α, β and h, $0 < \alpha, \beta, h < 1$ which only depend on B, λ and N such that for almost all $t \in (-\alpha r^2, 0)$,

$$mes\{\mathcal{N}_{t,h}\} \ge \frac{1}{11} mes\{K_{\beta r} \times S_{\beta r}\}.$$

Proof: Let

$$v = \ln^+ \left(\frac{1}{u + h^{\frac{9}{8}}}\right),$$

where h is a constant, $0 < h < 1$, to be determined later. Then v at points where v is positive, satisfies

$$\sum_{i,j=1}^{m_0} \partial_{x_i} (a_{ij}(x,t)\partial_{x_j} v) - (Dv)^T ADv + x^T BDv - \partial_t v = 0. \tag{3.1}$$

Let $\eta(x')$ be a smooth cut-off function so that

$$\eta(x') = 1, \quad \text{for} \quad |x'| < \beta r,$$

$$\eta(x') = 0, \quad \text{for} \quad |x'| \geq r.$$

Moreover, $0 \leq \eta \leq 1$ and $|D_{m_0}\eta| \leq \frac{2m_0}{(1-\beta)r}$.

Multiplying $\eta^2(x')$ to (3.1) and integrating by parts on $K_r \times S_{\beta r} \times (\tau, t)$

$$\begin{aligned} &\int_{K_{\beta r}} \int_{S_{\beta r}} v(t,x',\overline{x}) d\overline{x} dx' + \tfrac{1}{2\lambda} \int_\tau^t \int_{K_r} \int_{S_{\beta r}} \eta^2 \, |D_{m_0} v|^2 d\overline{x} dx' dt \\ &\leq \tfrac{C}{\beta^Q (1-\beta)^2} mes(S_{\beta r}) mes(K_{\beta r}) + \int_\tau^t \int_{K_r} \int_{S_{\beta r}} \eta^2 x^T BDv d\overline{x} dx' dt \\ &\quad + \int_{K_r} \int_{S_{\beta r}} v(\tau, x', \overline{x}) d\overline{x} dx', \qquad a.e. \quad \tau, t \in (-r^2, 0), \end{aligned} \tag{3.2}$$

where C only depends on λ, B and N. Let

$$I_B \equiv \int_{K_r} \int_{S_{\beta r}} \eta^2 \sum_{i,j=1}^{N} x_i b_{ij} \partial_{x_j} v d\overline{x} dx' = I_{B_1} + I_{B_2},$$

where

$$I_{B_1} = \int_{K_r} \int_{S_{\beta r}} \eta^2 \sum_{i=1}^{N} \sum_{j=1}^{m_0} x_i b_{ij} \partial_{x_j} v d\overline{x} dx',$$

$$I_{B_2} = \int_{K_r} \int_{S_{\beta r}} \eta^2 \sum_{i=1}^{N} \sum_{j=m_0+1}^{N} x_i b_{ij} \partial_{x_j} v d\overline{x} dx'.$$

On the other hand

$$
|I_{B_1}| \le \int_{K_r}\int_{S_{\beta r}} \varepsilon\eta^2|D_{m_0}v|^2 + C_\varepsilon\eta^2\sum_{j=1}^{m_0}\sum_{i=1}^{N}|x_ib_{ij}|^2 d\overline{x}dx'
$$

$$
\le \int_{K_r}\int_{S_{\beta r}} \varepsilon\eta^2|D_{m_0}v|^2 d\overline{x}dx' + C(\varepsilon,B,\lambda,N)\beta^{-Q}|K_{\beta r}||S_{\beta r}|, \tag{3.3}
$$

and

$$
\begin{aligned}
|I_{B_2}| &\le |\int_{K_r}\int_{S_{\beta r}} \eta^2\sum_{i=1}^{N}\sum_{j=m_0+1}^{N} x_ib_{ij}\partial_{x_j}v d\overline{x}dx'| \\
&\le |\int_{K_r}\int_{S_{\beta r}} -\eta^2\sum_{i=1}^{N}\sum_{j>m_0}\delta_{ij}b_{ij}v d\overline{x}dx'| \\
&\quad +|\int_{K_r}\int_{\partial_j S_{\beta r}} \eta^2\sum_{i=1}^{N}\sum_{j>m_0} x_ib_{ij}v d\overline{x_j}dx'| \\
&\le \lambda N\beta^{-Q}|K_{\beta r}||S_{\beta r}|\ln(h^{-\frac{9}{8}}) \\
&\quad +\lambda\sum_{i=1}^{N}\sum_{j>m_0}\tfrac{(\lambda N^2 r)^{\alpha_i}}{(\lambda N^2 r)^{\alpha_j}}\beta^{-2Q}|K_{\beta r}||S_{\beta r}|\ln(h^{-\frac{9}{8}}),
\end{aligned}
$$

where $\overline{x_j} = (x_{m_0+1},\dots,x_{j-1},x_{j+1},\dots,x_N)$. When $\alpha_i \ge \alpha_j$, we have

$$
\int_\tau^t |I_{B_2}| \le (\lambda N r^2 + \lambda r^2N^2)\beta^{-2Q}|K_{\beta r}||S_{\beta r}|\ln(h^{-\frac{9}{8}}),
$$

or $i<j$, thus $\alpha_j = \alpha_i + 2$ by the property of B, then

$$
\int_\tau^t |I_{B_2}| \le (\lambda N r^2 + \lambda^{-1}N^{-2})\beta^{-2Q}|K_{\beta r}||S_{\beta r}|\ln(h^{-\frac{9}{8}}).
$$

By $\lambda > 8$ choose r small enough, such that

$$
\lambda N r^2 + \lambda r^2N^2 + \lambda^{-1}N^{-2} < \frac{1}{8},
$$

thus

$$
\int_\tau^t |I_{B_2}| \le \frac{1}{4}\beta^{-2Q}|K_{\beta r}||S_{\beta r}|\ln(h^{-\frac{9}{8}}). \tag{3.4}
$$

Integrating by t to I_B, we have

$$
\int_\tau^t\int_{K_r}\int_{S_{\beta r}} \eta^2x^TBDv d\overline{x}dx'dt
$$

$$
\le \tfrac{1}{4}\beta^{-2Q}\ln(h^{-\frac{9}{8}})mes(S_{\beta r})mes(K_{\beta r}) \tag{3.5}
$$

$$
+\int_\tau^t\int_{K_r}\int_{S_{\beta r}} \varepsilon\eta^2|D_{m_0}v|^2 + C(\varepsilon,B,\lambda,N)\beta^{-Q}|K_{\beta r}||S_{\beta r}|.
$$

We shall estimate the measure of the set $\mathcal{N}_{t,h}$. Let

$$\mu(t) = mes\{(x', \overline{x})| \quad x' \in K_r,\, \overline{x} \in S_r,\, u(\cdot,t) \geq 1\}.$$

By our assumption, for $0 < \alpha < \frac{1}{2}$

$$\frac{1}{2}r^2 mes(S_r)mes(K_r) \leq \int_{-r^2}^{0} \mu(t)dt = \int_{-r^2}^{-\alpha r^2} \mu(t)dt + \int_{-\alpha r^2}^{0} \mu(t)dt,$$

that is

$$\int_{-r^2}^{-\alpha r^2} \mu(t)dt \geq (\frac{1}{2} - \alpha)r^2 mes(S_r)mes(K_r),$$

then there exists a $\tau \in (-r^2, -\alpha r^2)$, such that

$$\mu(\tau) \geq (\frac{1}{2} - \alpha)(1-\alpha)^{-1} mes(S_r)mes(K_r), \tag{3.6}$$

we have by noticing $v = 0$ when $u \geq 1$,

$$\int_{K_r}\int_{S_{\beta r}} v(\tau, x', \overline{x})d\overline{x}dx' \leq \frac{1}{2}(1-\alpha)^{-1} mes(S_r)mes(K_r)\ln(h^{-\frac{9}{8}}). \tag{3.7}$$

Now we choose $\varepsilon = \frac{1}{2\lambda}$ and α (near zero) and β (near one), so that

$$\frac{1}{4\beta^{2Q}} + \frac{1}{2\beta^{2Q}(1-\alpha)} \leq \frac{4}{5}. \tag{3.8}$$

By (3.2), (3.5), (3.7) and (3.8), and note the last term in (3.5) can be controlled by $C(B, \lambda, N)(1-\beta)^{-2}\beta^{-Q}|K_{\beta r}||S_{\beta r}|$, we deduce

$$\begin{aligned} &\textstyle\int_{K_{\beta r}}\int_{S_{\beta r}} v(t, x', \overline{x})d\overline{x}dx' \\ &\leq [2C(1-\beta)^{-2}\beta^{-Q} + \tfrac{4}{5}\ln(h^{-\frac{9}{8}})]mes(K_{\beta r} \times S_{\beta r}). \end{aligned} \tag{3.9}$$

When $(x', \bar{x}) \notin \mathcal{N}_{t,h}$, $u \geq h$, we have

$$\ln(\frac{1}{2h}) \leq \ln^+(\frac{1}{h + h^{\frac{9}{8}}}) \leq v,$$

then

$$\ln(\frac{1}{2h})mes(K_{\beta r} \times S_{\beta r} \setminus \mathcal{N}_{t,h}) \leq \int_{K_{\beta r}}\int_{S_{\beta r}} v(t, x', \overline{x})d\overline{x}dx'.$$

Since

$$\frac{C+\frac{4}{5}\ln(h^{-\frac{9}{8}})}{\ln(h^{-1})}\longrightarrow\frac{9}{10},\qquad \text{as}\quad h\to 0,$$

then there exists constant h_1 such that for $0<h<h_1$ and $t\in(-\alpha r^2,0)$

$$mes(K_{\beta r}\times S_{\beta r}\setminus\mathcal{N}_{t,h})\leq\frac{10}{11}mes(K_{\beta r}\times S_{\beta r}).$$

Then we proved our lemma.

Corollary 3.1 *Under the assumptions of Lemma 3.1, we can choose θ, $0<\theta<\alpha$ and $\theta<\beta$ small enough so that*

$$mes\{\mathcal{B}^-_{\beta r}\setminus\mathcal{B}^-_{\theta r}\cap\{(t,x)|\quad u\geq h\}\}\geq C_0(\alpha,\beta,\Lambda)mes\{\mathcal{B}^-_{\beta r}\},$$

where $0<C_0(\alpha,\beta,\Lambda)<1$.

Let $\chi(s)$ be a smooth function given by

$$\begin{array}{ll}\chi(s)=1 & if\quad s\leq\theta^{\frac{1}{2Q}}r,\\ \chi(s)=0 & if\quad s>r,\end{array}$$

where $\theta^{\frac{1}{2Q}}<\frac{1}{2}$ is a constant. Moreover, we assume that

$$0\leq-\chi'(s)\leq\frac{2}{(1-\theta^{\frac{1}{2Q}})r},$$

and $\chi'(s)<0$, if $\theta^{\frac{1}{2Q}}r<s<r$. Also for any β_1,β_2, with $\theta^{\frac{1}{2Q}}<\beta_1<\beta_2<1$, we have

$$|\chi'(s)|\geq C(\beta_1,\beta_2)>0,$$

if $\beta_1 r\leq s\leq\beta_2 r$.

For $x\in R^N$, $t<0$, we set

$$\mathcal{Q}=\{(x',\bar{x},t)|-r^2\leq t<0,\,x'\in K_{\frac{r}{\theta}},\,|x_j|\leq\frac{r^{\alpha_j}}{\theta},j=m_0+1,\cdots,N\},$$

$$\phi_0(x,t) = \chi([\theta^2|t|^Q\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle + \theta^2 \sum_{i=m_0+1}^N \frac{x_i^2}{r^{2\alpha_i - 2Q}} - C_1 t r^{2Q-2}]^{\frac{1}{2Q}}),$$

$$\phi_1(x) = \chi(\theta|x'|),$$

$$\phi(t,x) = \phi_0(t,x)\phi_1(x), \tag{3.10}$$

where $C_1 > 1$ is chosen so that

$$\begin{aligned} C_1 r^{2Q-2} \quad & \geq 2\theta^2|t|^Q|\langle x, Be^{tB}\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle| \\ & +\theta^2|t|^Q\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, A_0\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle \\ & +\theta^2|\textstyle\sum_{i=1}^N \sum_{j>m_0} 2x_i b_{ij} x_j r^{2Q-2\alpha_j}|, \end{aligned}$$

for all $z \in \mathcal{Q}$.

In the following discussion, $a \approx b$ means

$$C(B,\lambda,N)^{-1}a \leq b \leq C(B,\lambda,N)a.$$

With the notations given in section 2, for $s > 0$, or $t < 0$, we have following properties:

$$\mathcal{C}'(s) = A_0 - B^T\mathcal{C}(s) - \mathcal{C}(s)B, \tag{a}$$

$$Y\langle \mathcal{C}^{-1}(|t|)x, x\rangle = 4\langle x, B\mathcal{C}^{-1}(|t|)x\rangle - \langle \mathcal{C}^{-1}(|t|)x, A_0\mathcal{C}^{-1}(|t|)x\rangle, \tag{b}$$

$$\begin{aligned} Y\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle = \quad & 2\langle x, Be^{tB}\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle \\ & -\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, A_0\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle; \end{aligned} \tag{c}$$

(d) moreover, if $|t|$ is small enough, then

$$\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle \approx |D_{|t|^{-\frac{1}{2}}}x|^2, \tag{$d.1$}$$

$$(d.2) \qquad \langle \mathcal{C}^{-1}(|t|)e^{tB^T}B^Tx, e^{tB^T}x\rangle \leq C|t|^{-1}|D_{|t|^{-\frac{1}{2}}}x|^2,$$

$$(d.3) \qquad \langle A_0\mathcal{C}^{-1}(|t|)e^{tB^T}x, \mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle \leq C|t|^{-1}|D_{|t|^{-\frac{1}{2}}}x|^2,$$

where C depends on B, λ, and N.

The property (a) can be checked by the definition, in fact,

$$\mathcal{C}(s) = \int_0^s E(t)A_0E^T(t)\mathrm{dt},$$

then

$$\mathcal{C}'(s) = E(s)A_0E^T(s),$$

$$\mathcal{C}''(s) = E(s)(-B^T)A_0E^T(s) + E(s)A_0E^T(s)(-B) = -B^T\mathcal{C}'(s) - \mathcal{C}'(s)B,$$

integrating from 0 to s, we have

$$\mathcal{C}'(s) = A_0 - B^T\mathcal{C}(s) - \mathcal{C}(s)B.$$

To prove (b), we calculate

$$\begin{aligned}
&Y\langle \mathcal{C}^{-1}(|t|)x, x\rangle \\
&\quad = [\langle x, BD\rangle - \partial_t]\langle \mathcal{C}^{-1}(|t|)x, x\rangle \\
&\quad = 2\langle x, B\mathcal{C}^{-1}(|t|)x\rangle + \langle \partial_{|t|}\mathcal{C}^{-1}(|t|)x, x\rangle \\
&\quad = 2\langle x, B\mathcal{C}^{-1}(|t|)x\rangle - \langle \mathcal{C}^{-1}(|t|)\partial_{|t|}\mathcal{C}(|t|)\mathcal{C}^{-1}(|t|)x, x\rangle \\
&\quad = 4\langle x, B\mathcal{C}^{-1}(|t|)x\rangle - \langle \mathcal{C}^{-1}(|t|)x, A_0\mathcal{C}^{-1}(|t|)x\rangle.
\end{aligned}$$

The proof of (c) is the same as (b).
Applying (2.7) and (2.5),

$$\begin{aligned}
\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle &\approx \langle \mathcal{C}_0^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle \\
&= \langle \mathcal{C}_0^{-1}(1)D_{|t|^{-\frac{1}{2}}}e^{t\,B^T}x, D_{|t|^{-\frac{1}{2}}}e^{t\,B^T}x\rangle \\
&\approx \|e^{\tilde{B}}D_{|t|^{-\frac{1}{2}}}x\| \approx |D_{|t|^{-\frac{1}{2}}}x|^2
\end{aligned}$$

where $D_{|t|^{-\frac{1}{2}}}B^T = |t|^{-1}\tilde{B}D_{|t|^{-\frac{1}{2}}}$, $D_{|t|^{-\frac{1}{2}}}e^{tB^T} = e^{\tilde{B}}\,D_{|t|^{-\frac{1}{2}}}$ and $\tilde{B}$ has the form

$$\begin{pmatrix} |t|B_{0,0}^T & |t|^2B_{1,0}^T & \cdots & \cdots & |t|^{d+1}B_{d,0}^T \\ B_1^T & |t|B_{1,1}^T & \cdots & \cdots & |t|^dB_{d,1}^T \\ 0 & B_2^T & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_d^T & |t|B_{d,d}^T \end{pmatrix}$$

B is given by

$$\begin{pmatrix} B_{0,0} & B_1 & 0 & \cdots & 0 \\ B_{1,0} & B_{1,1} & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{d-1,0} & B_{d-1,1} & B_{d-1,2} & \cdots & B_d \\ B_{d,0} & B_{d,1} & B_{d,2} & \cdots & B_{d,d} \end{pmatrix}$$

then we obtain (d.1).

For any $x \in R^N$, by the Young inequality, (2.5) and (2.7), we have

$$\begin{aligned} & \langle \mathcal{C}^{-1}(|t|)B^Te^{tB^T}x, e^{tB^T}x\rangle \\ \leq\ & \varepsilon\langle \mathcal{C}^{-1}(|t|)B^Te^{tB^T}x, B^Te^{tB^T}x\rangle + \tfrac{1}{4\varepsilon}\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle \\ \leq\ & 2(\varepsilon\langle \mathcal{C}_0^{-1}(|t|)B^Te^{tB^T}x, B^Te^{tB^T}x\rangle + \tfrac{1}{4\varepsilon}\langle \mathcal{C}_0^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle) \\ \leq\ & C(B,\lambda,N)(\varepsilon|t|^{-2}|D_{|t|^{-\frac{1}{2}}}x|^2 + \tfrac{1}{4\varepsilon}|D_{|t|^{-\frac{1}{2}}}x|^2) \\ =\ & C(B,\lambda,N)|t|^{-1}|D_{|t|^{-\frac{1}{2}}}x|^2 \quad (\varepsilon = |t|), \end{aligned}$$

which is (d.2).

Let $y=e^{tB^T}x$, by (2.5), (2.6) and (2.7), we have

$$\begin{aligned}
&\langle y,\mathcal{C}^{-1}(|t|)A_0\mathcal{C}^{-1}(|t|)y\rangle\\
=\ &\langle y,\mathcal{C}^{-1}(|t|)A_0A_0\mathcal{C}^{-1}(|t|)y\rangle\\
=\ &|t|^{-1}\langle y,\mathcal{C}^{-1}(|t|)A_0D_{|t|^{\frac{1}{2}}}D_{|t|^{\frac{1}{2}}}A_0\mathcal{C}^{-1}(|t|)y\rangle\\
\leq\ &|t|^{-1}\langle y,\mathcal{C}^{-1}(|t|)D_{|t|^{\frac{1}{2}}}D_{|t|^{\frac{1}{2}}}\mathcal{C}^{-1}(|t|)y\rangle
\end{aligned}$$

consequently,

$$\begin{aligned}
&|t|^{-1}\langle y,\mathcal{C}^{-1}(|t|)D_{|t|^{\frac{1}{2}}}D_{|t|^{\frac{1}{2}}}\mathcal{C}^{-1}(|t|)y\rangle\\
\approx\ &|t|^{-1}\langle y,\mathcal{C}^{-1}(|t|)D_{|t|^{\frac{1}{2}}}\mathcal{C}_0(1)D_{|t|^{\frac{1}{2}}}\mathcal{C}^{-1}(|t|)y\rangle\\
=\ &|t|^{-1}\langle y,\mathcal{C}^{-1}(|t|)\mathcal{C}_0(|t|)\mathcal{C}^{-1}(|t|)y\rangle\\
\approx\ &|t|^{-1}\langle y,\mathcal{C}^{-1}(|t|)\mathcal{C}(|t|)\mathcal{C}^{-1}(|t|)y\rangle\\
=\ &|t|^{-1}\langle y,\mathcal{C}^{-1}(|t|)y\rangle\\
\approx\ &|t|^{-1}\langle y,\mathcal{C}_0^{-1}(|t|)y\rangle\\
\approx\ &|t|^{-1}|D_{|t|^{-\frac{1}{2}}}y|^2\\
\approx\ &|t|^{-1}|D_{|t|^{-\frac{1}{2}}}x|^2
\end{aligned}$$

and we obtain the proof of (d.3).

Remark 3.1 *By the definition of ϕ and the above arguments, it is easy to check that, for θ, r small and $t\leq 0$*
(1) $\phi(z)\equiv 1$, in $\mathcal{B}^-_{\theta r}$,
(2) $\mathrm{supp}\phi\subset\mathcal{Q}$,
(3) there exists $\alpha_1>0$, which depends on C_1, such that

$$\{(x,t)|-\alpha_1r^2\leq t<0, x'\in K_r,\bar{x}\in S_{\beta r}\}\subseteq \mathrm{supp}\phi,$$

(4) $0<\phi_0(z)<1$, *for* $z\in\{(x,t)|-\alpha_1 r^2\le t\le -\theta r^2, x'\in K_r, \bar{x}\in S_{\beta r}\}$.

Lemma 3.2 *Under the above notations, we have*
(e) $Y\phi_0(z)\le 0, \quad \text{for} \quad z\in\mathcal{Q}$*;*
(f)

$$|\int_{\mathcal{Q}}\phi_1|\langle A_0 D\phi_0, D\Gamma_1(z,\cdot)\rangle|-\int_{\mathcal{Q}}\phi_1\langle A_0 D\phi_0, D\Gamma_1(0,\cdot)\rangle|\le C_6\theta^2,$$

for $z\in\mathcal{B}^-_{\tilde{\theta}r}$, where C_6 is dependant on B, λ, N and $\tilde{\theta}$ depends on θ.

Proof:
Let

$$[\theta^2|t|^Q\langle\mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle+\theta^2\sum_{i=m_0+1}^{N}\frac{x_i^2}{r^{2\alpha_i-2Q}}-C_1tr^{2Q-2}]$$

be denoted by $[\cdots]$. Then

$$\begin{aligned}
Y\phi_0 &= \chi'([\cdots]^{\frac{1}{2Q}})\frac{1}{2Q}[\cdots]^{\frac{1}{2Q}-1}[\theta^2|t|^QY\langle\mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle\\
&+Q\theta^2|t|^{Q-1}\langle\mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle+C_1r^{2Q-2}\\
&+\theta^2\textstyle\sum_{i=1}^N\sum_{j>m_0}(2x_ib_{ij}x_jr^{2Q-2\alpha_j})]\\
&=\chi'([\cdots]^{\frac{1}{2Q}})\frac{1}{2Q}[\cdots]^{\frac{1}{2Q}-1}[\theta^2|t|^Q(2\langle x, Be^{tB}\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle\\
&-\langle\mathcal{C}^{-1}(|t|)e^{tB^T}x, A_0\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle)+Q\theta^2|t|^{Q-1}\langle\mathcal{C}^{-1}(|t|)e^{tB^T}x, e^{tB^T}x\rangle\\
&+C_1r^{2Q-2}+\theta^2\textstyle\sum_{i=1}^N\sum_{j>m_0}(2x_ib_{ij}x_jr^{2Q-2\alpha_j})].
\end{aligned}$$

We choose $C_1>1$, such that

$$\begin{aligned}
C_1r^{2Q-2}\ge\ &\theta^2|t|^Q(2|\langle x, Be^{tB}\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle|\\
&+\langle\mathcal{C}^{-1}(|t|)e^{tB^T}x, A_0\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle)\\
&+\theta^2|\textstyle\sum_{i=1}^N\sum_{j>m_0}2x_ib_{ij}x_jr^{2Q-2\alpha_j}|,
\end{aligned}$$

by the above (d),

$$\theta^2|t|^Q|\langle x, Be^{tB}\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle| \le C\theta^2|t|^{Q-1}|D_{|t|^{-\frac{1}{2}}}x|^2 \le Cr^{2Q-2},$$

for all $z \in \mathcal{Q}$.

Similar results holds for $\theta^2|t|^Q\langle \mathcal{C}^{-1}(|t|)e^{tB^T}x, A_0\mathcal{C}^{-1}(|t|)e^{tB^T}x\rangle$. For the term $x_ib_{ij}x_jr^{2Q-2\alpha_j}$, then either $\alpha_i \ge \alpha_j$ or $\alpha_j = \alpha_i + 2$, we also obtain

$$\theta^2|\sum_{i=1}^{N}\sum_{j>m_0} 2x_ib_{ij}x_jr^{2Q-2\alpha_j}| \le C(B,\lambda,N)r^{2Q-2}.$$

Thus $C_1(B,\lambda,N)$ is well defined, then $Y\phi_0(z) \le 0$ $(z \in \mathcal{Q})$ holds.

For the proof of (f), let $g(z) = \int_{\mathcal{Q}} \phi_1|\langle A_0D\phi_0, D\Gamma_1(z,\cdot)\rangle|(\zeta)$, then $g(z)$ is smooth and $g(z) \le g(0) + C(B,\lambda,N)|z|$. In fact,

$$\begin{aligned}
& g(0) \\
= \;& \textstyle\int_{\mathcal{Q}} \phi_1\langle A_0D\phi_0, D\Gamma_1(0,\cdot)\rangle \\
= \;& \textstyle\int_{\mathcal{Q}} \phi_1\chi'([\cdots]^{\frac{1}{2Q}})\frac{[\cdots]^{\frac{1}{2Q}-1}}{2Q}\theta^2|\tau|^Q\langle D_{m_0}\langle \mathcal{C}^{-1}(|\tau|)e^{\tau B^T}\xi, e^{tB^T}\xi\rangle,\, D_{m_0}^{(\zeta)}\Gamma_1(0,\cdot)\rangle \\
= \;& \textstyle\int_{\mathcal{Q}} \phi_1|\chi'([\cdots]^{\frac{1}{2Q}})|\frac{\theta^2|\tau|^Q\Gamma_1(0,\zeta)}{2Q[\cdots]^{1-\frac{1}{2Q}}}\langle e^{\tau B}\mathcal{C}^{-1}(|\tau|)e^{\tau B^T}\xi, A_0e^{\tau B}\mathcal{C}^{-1}(|\tau|)e^{\tau B^T}\xi\rangle.
\end{aligned}$$

We choose a domain $\mathcal{D}$ as in Remark 3.1, and $\mathcal{D} = \{(x,t)| - \alpha_1r^2 \le t \le -\frac{\alpha_1}{2}r^2, x' \in K_r, \bar{x} \in S_{\beta r}\}$, then by choosing small θ we get, $0 < \phi_0 < 1$, $\chi'([\cdots]^{\frac{1}{2Q}}) \approx r^{-1}$, $\phi_1 \equiv 1$, $[\cdots] \approx r^{2Q}$, and $\Gamma_1(0,\zeta) \approx |\tau|^{-\frac{Q}{2}}$ when $\zeta \in \mathcal{D}$. Hence

$$g(0) \ge C(B,\lambda,N)\theta^2r^{-Q}\int_{\mathcal{D}}\langle e^{\tau B}\mathcal{C}^{-1}(|\tau|)e^{\tau B^T}\xi, A_0e^{\tau B}\mathcal{C}^{-1}(|\tau|)e^{\tau B^T}\xi\rangle.$$

By $D_{|\tau|^{-\frac{1}{2}}}e^{\tau B^T} = e^{\tilde{B}}D_{|\tau|^{-\frac{1}{2}}}$ in $(d.1)$, and $D_{|\tau|^{\frac{1}{2}}}\mathcal{C}^{-1}(|\tau|)D_{|\tau|^{\frac{1}{2}}}$ which is positive and whose eigenvalues can be controlled by constants from (2.5) and (2.7),

then

$$r^2\langle e^{\tau B}\mathcal{C}^{-1}(|\tau|)e^{\tau B^T}\xi, A_0 e^{\tau B}\mathcal{C}^{-1}(|\tau|)e^{\tau B^T}\xi\rangle$$

$$= \ r^2|\tau|^{-1}\langle e^{\tilde{B}^T}D_{|\tau|^{\frac{1}{2}}}\mathcal{C}^{-1}(|\tau|)D_{|\tau|^{\frac{1}{2}}}e^{\tilde{B}}D_{|\tau|^{-\frac{1}{2}}}\xi, A_0 e^{\tilde{B}^T}D_{|\tau|^{\frac{1}{2}}}\mathcal{C}^{-1}(|\tau|)D_{|\tau|^{\frac{1}{2}}}e^{\tilde{B}}D_{|\tau|^{-\frac{1}{2}}}\xi\rangle,$$

which is positive and not dependent on r except zero measurable set, hence we get $g(0) \geq C_6\theta^2 > 0$ with C_6 as a constant dependant on B, λ, N. We can choose $\tilde{\theta}$ small, $0 < \tilde{\theta} < \theta$, such that $g(z) \leq g(0) + \frac{1}{2}C_6\theta^2$ for $z \in \mathcal{B}^-_{\tilde{\theta}r}$.

We now have the following Poincaré's type inequality.

Lemma 3.3 *Let w be a non-negative weak sub-solution of (1.2) in $\mathcal{B}^-_1$. Then there exists a constant C, only depends on B, λ and N, such that for $r < \theta < 1$*

$$\int_{\mathcal{B}^-_{\tilde{\theta}r}}(w(z) - I_0)^2_+ \leq C\theta^2 r^2 \int_{\mathcal{B}^-_{\frac{r}{\theta}}}|D_{m_0}w|^2, \tag{3.11}$$

where I_0 is given by

$$I_0 = max_{\mathcal{B}^-_{\tilde{\theta}r}}[I_1(z) + C_2(z)], \tag{3.12}$$

and

$$I_1(z) = \int_{\mathcal{B}^-_{\frac{r}{\theta}}}[\langle\phi_1 A_0 D\phi_0, D\Gamma_1(z,\cdot)\rangle w - \Gamma_1(z,\cdot)wY\phi](\zeta)d\zeta, \tag{3.13}$$

$$C_2(z) = \int_{\mathcal{B}^-_{\frac{r}{\theta}}}[\langle\phi_0 A_0 D\phi_1, D\Gamma_1(z,\cdot)\rangle w](\zeta)d\zeta,$$

where Γ_1 is the fundamental solution, and ϕ is given by (3.10).

Proof: We represent w in terms of the fundamental solution of Γ_1. For $z \in \mathcal{B}^-_{\theta r}$, we have

$$\begin{aligned} w(z) &= \textstyle\int_{\mathcal{B}^-_{\frac{r}{\theta}}}[\langle A_0 D(w\phi), D\Gamma_1(z,\cdot)\rangle - \Gamma_1(z,\cdot)Y(w\phi)](\zeta)d\zeta \\ &= I_1(z) + I_2(z) + I_3(z) + C_2(z), \end{aligned} \tag{3.14}$$

where $I_1(z)$ and $C_2(z)$ are given by (3.13) and

$$I_2(z) = \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [\langle (A_0 - A)Dw, D\Gamma_1(z,\cdot)\rangle\phi - \Gamma_1(z,\cdot)\langle ADw, D\phi\rangle](\zeta)d\zeta,$$

$$I_3(z) = \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [\langle ADw, D(\Gamma_1(z,\cdot)\phi)\rangle - \Gamma_1(z,\cdot)\phi Yw](\zeta)d\zeta.$$

From our assumption, w is a weak sub-solution of (1.2), and ϕ is a test function of this semi-cylinder. In fact, we let

$$\tilde{\chi}(\tau) = \begin{cases} 1 & \tau \leq 0, \\ 1 - n\tau & 0 \leq \tau \leq 1/n, \\ 0 & \tau \geq 1/n. \end{cases}$$

Then $\tilde{\chi}(\tau)\phi\Gamma_1(z,\cdot)$ can be a test function (see [2]). Let $n \to \infty$, we obtain $\phi\Gamma_1(z,\cdot)$ as a legitimate test function, and $I_3(z) \leq 0$. Then in $\mathcal{B}^-_{\theta r}$,

$$0 \leq (w(z) - I_0)_+ \leq I_2(z) = I_{21} + I_{22}.$$

By Corollary 2.1 we have

(3.15)

$$||I_{21}||_{L^2(\mathcal{B}^-_{\theta r})} \leq C(\lambda, N)\theta r||I_{21}||_{L^{2+\frac{4}{Q}}(\mathcal{B}^-_{\theta r})} \leq C(B, \lambda, N)\theta r||D_{m_0}w||_{L^2(\mathcal{B}^-_{\frac{r}{\theta}})}.$$

Similarly for I_{22},

$$||I_{22}||_{L^2(\mathcal{B}^-_{\theta r})} \leq |\mathcal{B}^-_{\theta r}|^{\frac{1}{2}-\frac{Q-2}{2Q+4}}||I_{22}||_{L^{2\tilde{k}}(\mathcal{B}^-_{\theta r})} \leq C(B, \lambda, N)\theta^2 r^2||D_{m_0}wD_{m_0}\phi||_{L^2(\mathcal{B}^-_{\frac{r}{\theta}})},$$

where $D_{m_0}\phi = \phi_0 D_{m_0}\phi_1 + \phi_1 D_{m_0}\phi_0$.

$$|\phi_0 D_{m_0}\phi_1| = |\phi_0\chi'(\theta|\xi'|)\theta D_{m_0}(|\xi'|)| \leq \frac{\theta}{r},$$

and

$$\begin{aligned} |\phi_1 D_{m_0}\phi_0| &\leq 2\phi_1|\chi'([\cdots]^{\frac{1}{2Q}})|\tfrac{1}{2Q}[\cdots]^{\frac{1}{2Q}-1}\theta^2|\tau|^Q|A_0 e^{\tau B}\mathcal{C}^{-1}(|t|)e^{\tau B^T}\xi| \\ &\leq C(B, \lambda, N)r^{-1}(\theta r^{2Q})^{\frac{1}{2Q}-1}\theta^2|\tau|^{Q-\frac{1}{2}}|D_{|\tau|^{-\frac{1}{2}}}\xi| \\ &\leq C(B, \lambda, N)r^{-1}(\theta r^{2Q})^{\frac{1}{2Q}-1}\theta^2 r^{2Q-1}\theta^{-1} \\ &\leq C(B, \lambda, N)\theta^{\frac{1}{2Q}}r^{-1}, \end{aligned}$$

thus

$$||I_{22}||_{L^2(\mathcal{B}^-_{\theta r})} \leq C(B,\lambda,N)\theta^2 r||D_{m_0}w||_{L^2(\mathcal{B}^-_{\frac{r}{\theta}})}.$$

Then we proved our lemma.

Now we apply Lemma 3.3 to the function

$$w = \ln^+ \frac{h}{u+h^{\frac{9}{8}}}.$$

If u is a weak solution of (1.2), obviously w is a weak sub-solution. We estimate the value of I_0 given by (3.12) and (3.13) in Lemma 3.3.

Lemma 3.4 *Under the assumptions of Lemma 3.3, there exist constants* λ_0, r_0 *and* $r_0 < \theta$. λ_0 *only depends on constants* α, β, λ, B, N, *and* ϕ, $0 < \lambda_0 < 1$, *such that for* $r < r_0$

$$|I_0| \leq \lambda_0 \ln(h^{-\frac{1}{8}}). \tag{3.16}$$

Proof: We first come to estimate $C_2(z)$ and often denote $x = (x', \bar{x}, t)$, and $\zeta = (\xi', \bar{\xi}, \tau)$. Note supp$\phi \in \mathcal{Q}$, and $z \in \mathcal{B}^-_{\theta r}$, then

(3.17)

$$\begin{aligned}
& |C_2(z)| \\
= \ & |\textstyle\int_{\mathcal{B}^-_{\frac{r}{\theta}}}[\langle\phi_0 A_0 D\phi_1, D\Gamma_1(z,\cdot)\rangle w](\zeta)d\zeta| \\
\leq \ & C(B,\lambda,N)\ln(h^{-\frac{1}{8}})\tfrac{2m_0\theta}{(1-\theta^{\frac{1}{2Q}})r}\sup_{\theta|\xi'|\geq\theta^{\frac{1}{2Q}}r}||\zeta^{-1}\circ z||^{-Q-1}\cdot\theta^{-N}|r|^{Q+2} \\
\leq \ & C(B,\lambda,N)\ln(h^{-\frac{1}{8}})\tfrac{2m_0}{(1-\theta^{\frac{1}{2Q}})r}\theta|\theta^{\frac{1}{2Q}-1}r-\theta r|^{-Q-1}\cdot\theta^{-N}|r|^{Q+2} \\
\leq \ & C(B,\lambda,N)\theta^{Q+\frac{3}{2}-N-\frac{1}{2Q}}\ln(h^{-\frac{1}{8}}) \\
= \ & C_3\theta^{\alpha_0}\ln(h^{-\frac{1}{8}})
\end{aligned}$$

where $\alpha_0 = Q + \frac{3}{2} - N - \frac{1}{2Q} > 0$ and similarly

$$
\begin{array}{rcl}
 & & |\int_{\mathcal{B}^-_{\frac{r}{\theta}}} [-\phi_0 Y\phi_1 \Gamma_1(z,\cdot)w](\zeta)d\zeta| \\
 & \leq & |\int_{\mathcal{B}^-_{\frac{r}{\theta}}} [-\phi_0 \chi'(\theta|\xi'|)\theta \sum_{i=1}^N \sum_{j=1}^{m_0} \xi_i b_{ij}\xi_j/|\xi'|\,\Gamma(z,\cdot)w](\zeta)d\zeta| \\
 & \leq & C(B,\lambda,N)|\theta^{\frac{1}{2Q}-1}r - \theta r|^{-Q}\theta^{-N}|r|^{Q+2}\ln(h^{-\frac{1}{8}}) \\
 & \leq & C(B,\lambda,N)\theta^{Q-N-\frac{1}{2}}r^2\ln(h^{-\frac{1}{8}}) \\
 & = & C_4\theta^{Q-N-\frac{1}{2}}r^2\ln(h^{-\frac{1}{8}}) \\
 & \leq & C_4\theta^{\tilde{\alpha_0}}\ln(h^{-\frac{1}{8}})
\end{array}
\tag{3.18}
$$

where $\tilde{\alpha_0} = Q + \frac{3}{2} - N > 0$, if $r < \theta$.

Now we let $w \equiv 1$ then (3.14) gives, for $z \in \mathcal{B}^-_{\theta r}$,

$$
\begin{array}{rcl}
1 & = & \int_{\mathcal{B}^-_{\frac{r}{\theta}}} [\langle \phi_1 A_0 D\phi_0, D\Gamma_1(z,\cdot)\rangle - \phi_1\Gamma_1(z,\cdot)Y\phi_0](\zeta)d\zeta \\
 & & - \int_{\mathcal{B}^-_{\frac{r}{\theta}}} \phi_0\Gamma_1(z,\cdot)Y\phi_1(\zeta)d\zeta + C_2(z)|_{w=1},
\end{array}
\tag{3.19}
$$

where ϕ is given by (3.10). By Lemma 3.2, for $z \in B^-_{\tilde{\theta} r}$,

$$
-\phi_1\Gamma_1(z,\cdot)Y\phi_0 \geq 0. \tag{3.20}
$$

$$
g(z) = \int_{\mathcal{Q}} |\langle \phi_1 A_0 D\phi_0, D\Gamma_1(z,\cdot)\rangle| \leq g(0) + \frac{1}{2}C_6\theta^2. \tag{3.21}
$$

We only need to prove $-\phi_1\Gamma_1(z,\cdot)Y\phi_1$ has a positive lower bound in a domain which w vanishes, and this bound independent of r and small θ. So we can find a λ_0, $0 < \lambda_0 < 1$, such that this lemma holds and λ_0 is independent of r and small θ. We observe that the support of $\chi'(s)$ is in the region $\theta^{\frac{1}{2Q}}r < s < r$, thus for some $\beta' < 1$ (we choose β' near one), the set $\mathcal{B}^-_{\beta' r} \setminus \mathcal{B}^-_{\sqrt{\theta} r}$ with $|t| > \theta r^2/C_1$ is contained in the support of ϕ' . Then we can prove that the integral of (3.20) on the domain $\mathcal{B}^-_{\beta' r} \setminus \mathcal{B}^-_{\sqrt{\theta} r}$ with $|t| > \theta r^2/C_1$ is lower bounded by a positive constant.

For $z \in B^-_{\theta r}$, $0 < \alpha_1 \leq \alpha$ and set

$$
\zeta \in Z = \{(\xi,\tau) | -\alpha_1 r^2 \leq \tau \leq -\frac{\alpha_1}{2}r^2,\ x' \in K_r,\ \bar{x} \in S_{\beta r},\ w(\xi,\tau) = 0\},
$$

then $|Z| = C(\alpha_1, \lambda, N)r^{Q+2}$ by Lemma 3.1 and Corollary 3.1. We note that $w(\zeta) = 0$, $\phi_1(\zeta) = 1$, $|\chi'([\cdots]^{\frac{1}{2Q}})| \geq C(\alpha_1, B, \lambda, N) > 0$ when $\zeta \in Z$ and θ is small, then

$$
\begin{aligned}
&\textstyle\int_Z [-\phi_1 \Gamma_1(z,\cdot) Y \phi_0](\zeta)\, d\zeta \\
&= -\textstyle\int_Z \phi_1 \Gamma_1(z,\cdot) \chi'([\cdots]^{\frac{1}{2Q}}) \frac{1}{2Q} [\cdots]^{\frac{1}{2Q}-1} [\theta^2 |\tau|^Q (2\langle \xi, Be^{\tau B} \mathcal{C}^{-1}(|\tau|) e^{\tau B^T} \xi\rangle \\
&- \langle \mathcal{C}^{-1}(|\tau|) e^{\tau B^T} \xi, A_0 \mathcal{C}^{-1}(|\tau|) e^{\tau B^T} \xi\rangle) + Q\theta^2 |\tau|^{Q-1} \langle \mathcal{C}^{-1}(|\tau|) e^{\tau B^T} \xi, e^{\tau B^T} \xi\rangle \\
&+ C_1 r^{2Q-2} + \theta^2 \textstyle\sum_{i=1}^N \sum_{j>m_0} (2\xi_i b_{ij} \xi_j r^{2Q-\alpha_j})] d\zeta \\
&\geq C(B, \lambda, \alpha, N) \textstyle\int_Z r^{2Q-2} [r^{2Q}]^{\frac{1}{2Q}-1} r^{-1} \Gamma_1(\zeta^{-1} \circ z; 0) d\zeta \\
&\geq C(B, \lambda, \alpha, N) \textstyle\int_Z r^{-2} (t-\tau)^{-\frac{Q}{2}} \exp(-C|D_{|t-\tau|^{-\frac{1}{2}}} (x - E(t-\tau)\xi)|^2) \\
&\geq C(B, \lambda, \alpha, N) \textstyle\int_Z r^{-2} (t-\tau)^{-\frac{Q}{2}} \exp(-C|D_{|\tau|^{-\frac{1}{2}}} \xi|^2) \quad \text{(the same as(d.1))} \\
&\geq C(B, \lambda, \alpha, N) \textstyle\int_Z r^{-Q-2} \\
&= C(B, \lambda, \alpha, \beta, N) = C_5 > 0.
\end{aligned}
$$

Similarly we can choose a small data, still denote $\tilde{\theta}$, such that

$$\int_{\mathcal{Q}} [-\phi_1 \Gamma_1(z,\cdot) Y \phi_0](\zeta)\, d\zeta \leq \int_{\mathcal{Q}} [-\phi_1 \Gamma_1(0,\cdot) Y \phi_0](\zeta)\, d\zeta + \frac{1}{2} C_6 \theta^2,$$

for $z \in B^-_{\tilde{\theta} r}$. We can choose a small θ ,then choose $\tilde{\theta}$ small, and fixed them

from now on, $r_0 < \theta$, such that

$$
\begin{aligned}
& |I_0| \\
\leq\; & (\textstyle\int_{\mathcal{Q}} |\langle \phi_1 A_0 D\phi_0, D\Gamma_1(z,\cdot)\rangle| + [-\phi_1\Gamma_1(z,\cdot)Y\phi_0](\zeta)\, d\zeta - C_5)\ln(h^{-\frac{1}{8}}) \\
& +(C_3\theta^{\alpha_0} + C_4\theta^{\tilde{\alpha_0}})\ln(h^{-\frac{1}{8}}) \\
\leq\; & (1 - C_5 + C_6\theta^2 + C_3\theta^{\alpha_0} + C_4\theta^{\tilde{\alpha_0}})\ln(h^{-\frac{1}{8}}) + (C_3\theta^{\alpha_0} + C_4\theta^{\tilde{\alpha_0}})\ln(h^{-\frac{1}{8}}) \\
\leq\; & \lambda_0 \ln(h^{-\frac{1}{8}}).
\end{aligned}
$$

Where $0 < r < r_0$, $0 < \lambda_0 < 1$, depends on α, β, B, λ, N, and ϕ.

Lemma 3.5 *Suppose that* $u(x,t) \geq 0$ *be a solution of equation (1.2) in* $\mathcal{B}_r^-$ *centered at* $(0,0)$ *and*

$$mes\{(x,t) \in \mathcal{B}_r^-, \quad u \geq 1\} \geq \frac{1}{2} mes(\mathcal{B}_r^-).$$

Then there exist constant θ *and* h_0*,* $0 < \theta, h_0 < 1$ *which only depend on* B*,* λ*,* λ_0 *and* N *such that*

$$u(x,t) \geq h_0 \quad in \quad \mathcal{B}_{\theta r}^-.$$

Proof: We consider

$$w = \ln^+\left(\frac{h}{u + h^{\frac{9}{8}}}\right),$$

for $0 < h < 1$, to be decided. By applying Lemma 3.3 to w, we have

$$\fint_{\mathcal{B}_{\theta r}^-} (w - I_0)_+^2 \leq C(B,\lambda,N)\frac{\theta r^2}{|\mathcal{B}_{\theta r}^-|}\int_{\mathcal{B}_r^-} |D_{m_0} w|^2.$$

Let $\tilde{u} = \frac{u}{h}$, then $\tilde{u}$ satisfies the conditions of Lemma 3.1. We can get similar

estimates as (3.2), (3.5), (3.7) and (3.8), hence we have
(3.22)

$$C(B,\lambda,N)\tfrac{\theta r^2}{|\mathcal{B}^-_{\theta r}|}\int_{\mathcal{B}^-_r}|D_{m_0}w|^2$$

$$\le C(B,\lambda,N)\tfrac{\theta r^2}{|\mathcal{B}^-_{\theta r}|}[C(B,\lambda,N)(1-\beta)^{-2}\beta^{-Q}+\tfrac{4}{5}\ln(h^{-\frac{1}{8}})]mes(K_{\beta r}\times S_{\beta r})$$

$$\le C(\theta,B,N,\lambda)\ln(h^{-\frac{1}{8}}),$$

where θ has been chosen. By Lemma 2.2, there exists a constant, still denoted by θ, such that for $z\in\mathcal{B}^-_{\theta r}$,

$$w-I_0\le C(B,\lambda,N)(\ln(h^{-\frac{1}{8}}))^{\frac{1}{2}}. \tag{3.23}$$

Therefore we may choose h_0 small enough, so that

$$C(\ln(\frac{1}{h_0^{\frac{1}{8}}}))^{\frac{1}{2}}\le\ln(\frac{1}{2h_0^{\frac{1}{8}}})-\lambda_0\ln(\frac{1}{h_0^{\frac{1}{8}}}).$$

Then (3.16) and (3.23) implies

$$\max_{\mathcal{B}^-_{\theta r}}\frac{h_0}{u+h_0^{\frac{9}{8}}}\le\frac{1}{2h_0^{\frac{1}{8}}},$$

which implies $\min_{\mathcal{B}^-_{\theta r}}u\ge h_0^{\frac{9}{8}}$, then we finished the proof of this Lemma.

Proof of Theorem 1.1. We may assume that $M=\max_{\mathcal{B}^-_r}(+u)=\max_{\mathcal{B}^-_r}(-u)$, otherwise we replace u by $u-c$, since u is bounded locally. Then either $1+\frac{u}{M}$ or $1-\frac{u}{M}$ satisfies the assumption of Lemma 3.5, and we suppose $1+\frac{u}{M}$ does it, thus Lemma 3.5 implies existing $h_0>0$ such than $\inf_{\mathcal{B}^-_{\theta r}}(1+\frac{u}{M})\ge h_0$, i.e. $u\ge M(h_0-1)$, then

$$Osc_{\mathcal{B}^-_{\theta r}}u\le M-M(h_0-1)\le(1-\frac{h_0}{2})Osc_{\mathcal{B}^-_r}u,$$

which implies the C^α regularity of u near point $(0,0)$ by the standard iteration arguments. By the left invariant translation group action, we know that u is C^α in the interior.

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